Example 12 Find the conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

Solution We have, $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$

$$= \frac{6+9i-4i+6}{2-i+4i+2} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i}$$

$$= \frac{48-36i+20i+15}{16+9} = \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i$$

Therefore, conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ is $\dfrac{63}{25} + \dfrac{16}{25}i$.

Example 13 Find the modulus and argument of the complex numbers:

(i) $\frac{1+i}{1-i}$, (ii) $\frac{1}{1+i}$

Solution (i) We have, $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1-1+2i}{1+1} = i = 0 + i$

Now, let us put $0 = r\cos\theta$, $1 = r\sin\theta$

Squaring and adding, $r^2 = 1$ i.e., $r = 1$ so that

$$\cos\theta = 0,\ \sin\theta = 1$$

Therefore, $\theta = \frac{\pi}{2}$

Hence, the modulus of $\frac{1+i}{1-i}$ is 1 and the argument is $\frac{\pi}{2}$.

(ii) We have $$\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{1+1} = \frac{1}{2} - \frac{i}{2}$$

Let $$\frac{1}{2} = r\cos\theta,\ -\frac{1}{2} = r\sin\theta$$

Proceeding as in part (i) above, we get $r = \frac{1}{\sqrt{2}}$; $\cos\theta = \frac{1}{\sqrt{2}}$, $\sin\theta = \frac{-1}{\sqrt{2}}$

Therefore $$\theta = \frac{-\pi}{4}$$

Hence, the modulus of $\frac{1}{1+i}$ is $\frac{1}{\sqrt{2}}$, argument is $\frac{-\pi}{4}$.

Example 14 If $x + iy = \frac{a+ib}{a-ib}$, prove that $x^2 + y^2 = 1$.

Solution We have,

$$x + iy = \frac{(a+ib)(a+ib)}{(a-ib)(a+ib)} = \frac{a^2-b^2+2abi}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2} + \frac{2ab}{a^2+b^2}i$$

So that, $x - iy = \dfrac{a^2-b^2}{a^2+b^2} - \dfrac{2ab}{a^2+b^2}i$

Therefore,

$$x^2 + y^2 = (x+iy)(x-iy) = \frac{(a^2-b^2)^2}{(a^2+b^2)^2} + \frac{4a^2b^2}{(a^2+b^2)^2} = \frac{(a^2+b^2)^2}{(a^2+b^2)^2} = 1$$

Example 15 Find real θ such that

$$\frac{3+2i\sin\theta}{1-2i\sin\theta} \text{ is purely real.}$$

Solution We have,

$$\frac{3+2i\sin\theta}{1-2i\sin\theta} = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$$

$$= \frac{3+6i\sin\theta+2i\sin\theta-4\sin^2\theta}{1+4\sin^2\theta} = \frac{3-4\sin^2\theta}{1+4\sin^2\theta} + \frac{8i\sin\theta}{1+4\sin^2\theta}$$

We are given the complex number to be real. Therefore

$$\frac{8\sin\theta}{1+4\sin^2\theta} = 0, \text{ i.e., } \sin\theta = 0$$

Thus $\theta = n\pi,\ n \in \mathbf{Z}$.

Example 16 Convert the complex number $z = \dfrac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ in the polar form.

Solution We have, $z = \dfrac{i-1}{\frac{1}{2} + \frac{\sqrt{3}}{2}i}$

$$= \frac{2(i-1)}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{2\left(i+\sqrt{3}-1+\sqrt{3}i\right)}{1+3} = \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2}i$$

Now, put $\dfrac{\sqrt{3}-1}{2} = r\cos\theta,\ \dfrac{\sqrt{3}+1}{2} = r\sin\theta$

Squaring and adding, we obtain

$$r^2 = \left(\frac{\sqrt{3}-1}{2}\right)^2 + \left(\frac{\sqrt{3}+1}{2}\right)^2 = \frac{2\left(\left(\sqrt{3}\right)^2+1\right)}{4} = \frac{2\times 4}{4} = 2$$

Hence, $r = \sqrt{2}$ which gives $\cos\theta = \frac{\sqrt{3}-1}{2\sqrt{2}}, \quad \sin\theta = \frac{\sqrt{3}+1}{2\sqrt{2}}$

Therefore, $\theta = \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{12}$ (Why?)

Hence, the polar form is

$$\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

Summary

- A number of the form $a + ib$, where a and b are real numbers, is called a *complex number*, a is called the *real part* and b is called the *imaginary* part of the complex number.
- Let $z_1 = a + ib$ and $z_2 = c + id$. Then
  (i) $z_1 + z_2 = (a + c) + i\,(b + d)$
  (ii) $z_1 z_2 = (ac - bd) + i\,(ad + bc)$
- For any non-zero complex number $z = a + ib$ $(a \neq 0, b \neq 0)$, there exists the complex number $\frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$, denoted by $\frac{1}{z}$ or z^{-1}, called the *multiplicative inverse* of z such that $(a + ib)\left(\frac{a^2}{a^2+b^2} + i\frac{-b}{a^2+b^2}\right) = 1 + i0 = 1$
- For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$
- The conjugate of the complex number $z = a + ib$, denoted by $\bar{z}$, is given by $\bar{z} = a - ib$.
- The polar form of the complex number $z = x + iy$ is $r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$ (the modulus of z) and $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$. (θ is known as the argument of z. The value of θ, such that $-\pi < \theta \leq \pi$, is called the *principal argument* of z.
- A polynomial equation of n degree has n roots.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in R$, $a \neq 0$, $b^2 - 4ac < 0$, are given by $x = \frac{-b \pm \sqrt{4ac - b^2}\,i}{2a}$.